International Journal of Theoretical Physics, Vol. 10, No. 3 (1974), pp. 145–151

Zero Value of the Schwarzschildian Mass of Asymptotically Euclidian Time-Symmetrical Gravitational Waves

V. N. FOLOMESHKIN

Institute of High Energy Physics, Serpukhov, Moscow District, U.S.S.R.

Received 12 *June* 1973

Abstract

It is shown that a coordinate system with simple coordinate conditions can be chosen such that one can explicitly see that the Schwarzschildian mass of an asymptotically Euclidian time-symmetrical system of gravitational waves is equal to zero. It is explicitly seen in the coordinate system with coordinate conditions $\partial_i(-gg^{ik}) = 0$ and in the set of coordinate systems with the coordinate condition $\partial_i(-gg^{0i}) = 0$. In this set of coordinate systems one of the field equations can be written in the form $-8\pi\sqrt{(-g)}(T_0{}^0 - \frac{1}{2}T) = \partial_\alpha L_0^{0\alpha}$ where $L_i^{kn} = \frac{1}{2}\sqrt{(-g)}(g^{mn}\Gamma_{im}^k - g^{km}\Gamma_{im}^n)$, $\alpha = 1, 2, 3$. From this equation it follows that $m = 2\int (T_0{}^0 - \frac{1}{2}T)\sqrt{(-g)}\, dV$, and $m = 0$ at $T_i{}^k = 0$.

1. *Introduction*

The problem of the energy of gravitational waves has been discussed in many papers. For example, the following results have been obtained: (1) gravitational waves have a negative energy (Hu, 1947; Peres, 1959; Havas & Goldberg, 1962; Sexl, 1966; Petrov, Piragas & Dobrovolsky, 1968); (2) the energy of gravitational waves is equal to zero (Brdicka, 1951; Infeld & Sheidegger, 1951; Sheidegger, 1951, 1953, 1955; Infeld, 1953, 1956, 1959; Infeld & Plebansky, 1960; Rosen, 1956; Weber & Wheeler, 1957; Møller, 1958; Trautman, 1958; Dirac, 1959; Capella, 1961; Pirani, 1961; Misner, 1963; Cahen & Sengier-Diels, 1963; Langer, 1963; Kuchar & Langer, 1963; Shirokov, 1971, 1972; Folomeshkin, 1970; Vlasov, 1971); (3) gravitational waves have positive energy (Araki, 1959; Brill, 1959; Arnowitt, Deser & Misner, 1960; Trautman, 1958; Komar, 1963; Brill, Deser & Fadeev, 1968; Brill & Deser, 1968); (4) the energy of gravitational waves is indefinite (Vu Thanh Khiet, 1965).

In most papers the energy of the gravitational waves was determined with the generalised complex ('pseudotensor') of the 'energy-momentum'. It is known that the generalised complex has no relation to the usual notion of the energy-momentum in the general case and to the correct covariant formulation of the energy-momentum problem (Folomeshkin, 1967, 1969, 1971, 1972). The results obtained with the generalised complex have no definite physical sense and their disagreement is not astonishing. This fact is known from the first years of existence of General Relativity (Lorentz, 1916a, 1916b; Schrödinger, 1918; Bauer, 1918; Møller, 1964).

The determination of the Schwarzschildian mass of a closed, finite, asymptotically Euclidian system of gravitational waves according to the asymptotic gravitational field of this system has more definite sense (Araki, 1959; Brill, 1959; Arnowitt, Deser & Misner, 1960; Brill, Deser & Fadeev, 1968; Brill & Deser, 1968). For a simple case of the time symmetrical gravitational waves this problem was examined by Brill (1959). He showed that if the mass of the t-symmetrical asymptotically Euclidian system is different from zero then it will be positive. But can the mass of such a system be different from zero at $T_i^k = 0$?

The present paper shows that a coordinate system with simple coordinate conditions can be chosen such that one can explicitly see the Schwarzschildian mass of the asymptotically Euclidian t-symmetrical system of the gravitational waves is equal to zero when $T_i^k = 0$.

2. *Illustration of the Brill's Result*

The essence of Brill's result (Brill, 1959) can be easily seen after the simplest example of a statical spherically symmetrical system. Let the metrics of such a system have the form

$$dS^2 = e^{\nu}\, dt^2 - e^{\mu}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) \tag{2.1}$$

The equation for the T_0^0 component has the form

$$-8\pi T_0^0 e^{\mu} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\mu}{\partial r}\right) + \tfrac{1}{4}(\partial\mu/\partial r)^2 \tag{2.2}$$

At $r \to \infty$, $e^{\mu} \to (1 + m/(2r))^4$.

Let us integrate (2.2) over three-dimensional volume. We obtain

$$m = -\frac{1}{2}\int \frac{\partial}{\partial r}\left(r^2 \frac{\partial\mu}{\partial r}\right) dr = \int \left[T_0^0 e^{\mu} + \frac{1}{32\pi}(\partial\mu/\partial r)^2\right] d^3X \tag{2.3}$$

We see that at $T_0^0 = 0$ the value of the Schwarzschildian mass is not negative. This is Brill's result. (Brill considered t-symmetrical an axially symmetrical system.)

If besides the equation for T_0^0 we take into account the equations for other components of T_i^k then the following expression is known to be for the Schwarzschildian mass of the static system (Nordström, 1918; Tolman, 1930; Whittaker, 1935; Papapetrou, 1947)

$$m = 2 \int (T_0^{\ 0} - \tfrac{1}{2}T)\sqrt{(-g)}\, dV \tag{2.4}$$

From (2.4) it follows directly that

$$m = 0 \quad \text{when} \quad T_i^{\ k} = 0 \tag{2.5}$$

Result (2.5) can also be obtained in another way. Let us use the coordinate system with the metrics

$$dS^2 = e^{\lambda}\, dt^2 - e^{\mu}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2.6}$$

Then from the equation for $T_0^{\ 0}$

$$-8\pi T_0^{\ 0} = \frac{1}{r^2}\frac{\partial}{\partial r}\,[r(e^{-\mu} - 1)] \tag{2.7}$$

it follows that

$$m = \int T_0^{\ 0}\,(4\pi r^2\, dr) \tag{2.8}$$

and, consequently, condition (2.5) is valid.

This simplest illustrative example shows that to stabilise Brill's result we should try: (1) to use other field equations in addition to the equation for $T_0^{\ 0}$; (2) to use a coordinate system in which we can explicitly see the dependence of the asymptotical metrical tensor on the components of $T_i^{\ k}$.

3. *The Choice of the Coordinate System*

As is known the scalar curvature R can be divided into two parts

$$R = G + \frac{1}{\sqrt{(-g)}}\,\partial_k \omega^k \tag{3.1}$$

where the quantities G and ω^k have the form

$$G = g^{pm}(\Gamma^n_{ps}\Gamma^s_{mn} - \Gamma^s_{ns}\Gamma^n_{pm}), \tag{3.2}$$

$$\omega^k = \sqrt{(-g)}(g^{mn}\Gamma^k_{mn} - g^{kn}\Gamma^m_{nm}) \tag{3.3}$$

One of the known coordinate conditions is the harmonic coordinate condition (De Donder, 1921; Lanczos, 1923; Fok, 1961)

$$\partial_k(\sqrt{(-g)}g^{ik}) = 0 \tag{3.4}$$

Other possible coordinate conditions are the following

$$\partial_k(-gg^{ik}) = 0 \tag{3.5}$$

In the coordinate system which satisfies condition (3.5) the scalar curvature R does not contain second derivatives of the metrical tensor and takes a simple form

$$R = G \tag{3.6}$$

In this coordinate system the generalised Einstein (1916) complex

$$\theta_i^{\ k} = \frac{1}{16\pi}\left(G\delta_i^{\ k} - \frac{\partial G}{\partial g_k^{mn}} g_i^{mn}\right) \tag{3.7}$$

coincides with the generalised Lorentz (1916) complex

$$\lambda_i^{\ k} = \frac{1}{16\pi}\left\{R\delta_i^{\ k} - \left[\frac{\partial R}{\partial g_k^{mn}} - \left(\frac{\partial R}{\partial g_{k,l}^{mn}}\right)_{,l}\right] g_i^{mn} - \frac{\partial R}{\partial g_{k,l}^{mn}} g_{i,l}^{mn}\right\} \tag{3.8}$$

(here $g_k^{mn} = \partial_k g^{mn}$), and Freud's (1939) superpotential coincides with Møller's (1958a, b) superpotential. Therefore the coordinate system which satisfies condition (3.5) possesses a number of important advantages (at least from the theoretical point of view).

Let us consider the problem of the Schwarzschildian mass of the t-symmetrical system of gravitational waves in this coordinate system.

4. *Mass of T-Symmetrical Gravitational Waves*

In the coordinate system which satisfies condition (3.5) the gravitational field equations

$$8\pi T_i^{\ k} = R_i^{\ k} - \tfrac{1}{2}\delta_i^{\ k} R \tag{4.1}$$

take the form

$$8\pi T_i^{\ k} = -\frac{1}{\sqrt{(-g)}}\partial_n L_i^{kn} - 8\pi\theta_i^{\ k} \tag{4.2}$$

where the superpotential L_i^{kn} is

$$L_i^{kn} = \tfrac{1}{2}\sqrt{(-g)}(g^{mn}\Gamma_{im}^k - g^{km}\Gamma_{im}^n) \tag{4.3}$$

Bearing in mind result (2.4) let us write equation (4.2) in the form

$$-8\pi(T_i^{\ k} - \tfrac{1}{2}\delta_i^{\ k} T) = \frac{1}{\sqrt{-g}}\partial_n L_i^{kn} - \frac{1}{2}\frac{\partial G}{\partial g_k^{mn}} g_i^{mn} \tag{4.4}$$

and integrate the equation for the $K_0^{\ 0}$ component over the three-dimensional volume. We then obtain

$$m = -\frac{1}{4\pi}\int \partial_\alpha L_0^{0\alpha}\, dV$$

$$= 2\int (T_0^{\ 0} - \tfrac{1}{2}T)\sqrt{(-g)}\, dV - \frac{1}{8\pi}\int \frac{\partial G}{\partial g_0^{mn}} g_0^{mn}\sqrt{(-g)}\, dV \tag{4.5}$$

where $\alpha = 1, 2, 3$. For the t-symmetrical case when

$$g_0^{mn} = 0 \tag{4.6}$$

equality (4.5) takes the form (2.4) from which (2.5) follows.

Result (2.5) can also be obtained at less stringent restrictions than (3.5) for the coordinate system. Instead of four conditions (3.5) let us demand that our coordinate system should satisfy one coordinate condition

$$\partial_k(-gg^{0k}) = 0 \tag{4.7}$$

For the time-orthogonal metric condition (4.7) is equivalent to the condition

$$g^{\alpha\beta}\partial_0 g_{\alpha\beta} = 0 \tag{4.8}$$

If, for example, the metric has the orthogonal form

$$dS^2 = e^\nu\, dt^2 - e^\lambda\, dr^2 - e^\mu r^2\, d\theta^2 - e^\sigma z^2 \sin^2\theta\, d\varphi^2 \tag{4.9}$$

or

$$dS^2 = e^\nu\, dt^2 - e^\lambda\, d\rho^2 - e^\mu\, dz^2 - e^\sigma \rho^2\, d\varphi^2 \tag{4.10}$$

then coordinate condition (4.8) is satisfied deliberately, for example, at

$$\lambda + \mu + \sigma = 0 \tag{4.11}$$

The latter condition is more stringent than (4.8) but in most practical cases (e.g. for all axially symmetric metrics (4.10)) it is a permissible coordinate condition.

Using the simple coordinate transformation

$$r = \bar{r}\exp(-\mu/4) \tag{4.12}$$

where

$$\exp(\mu) = 1 + 2m/\bar{r} + \tfrac{15}{2}(m/\bar{r})^2 + \cdots \tag{4.13}$$

the usual Schwarzschildian metric (2.6) is transformed to the form

$$dS^2 = e^\nu\, dt^2 - e^\mu\, d\bar{r}^2 - e^{-\mu/2}\bar{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{4.14}$$

which satisfies condition (4.11). In this case

$$e^\nu = 1 - 2m/\bar{r} - (m/\bar{r})^2 - 3(m/\bar{r})^3 - \cdots \tag{4.15}$$

In the arbitrary coordinate system field equations (4.1) can be written in the form

$$8\pi\sqrt{(-g)}(T_i^k - \tfrac{1}{2}\delta_i^k T) = -\partial_n L_i^{kn} + \frac{1}{2}\left(\partial_i\omega^k + \frac{\partial\sqrt{(-g)}G}{\partial g_k^{mn}}\, g_i^{mn}\right) \tag{4.16}$$

Having integrated T_0^0 equation, we obtain the expression for the Schwarzschildian mass of the arbitrary asymptotically Euclidian system

$$m = 2\int (T_0^0 - \tfrac{1}{2}T)\sqrt{(-g)}\, dV - \frac{1}{8\pi}\int \left(\partial_0 \omega^0 + \frac{\partial \sqrt{(-g)}G}{\partial g_0^{mn}} g_0^{mn}\right) dV \quad (4.17)$$

For the t-symmetrical gravitational waves in the coordinate system which satisfy condition (4.7), equality (4.17) takes the form (2.4), and we obtain relation (2.5).

5. *Conclusion*

The fact that the Schwarzschildian mass of the t-symmetrical asymptotically Euclidian gravitational waves is equal to zero never means that the space is flat in the region considered or that the gravitational waves are undetectable. Coordinate conditions (3.5) and (4.7) do not demand, by no means, realisation of the condition $R_{iklm} = 0$. In the experiment we determine a mechanical effect of the gravitational waves on the detector. This effect is simply determined with the curvature tensor and has no direct relation to the Schwarzschildian mass of any closed system of the gravitational waves. This problem has been sufficiently discussed, for example by Weber & Wheeler (1957), who have shown that the cylindrical waves (Einstein & Rosen, 1937; Rosen, 1953, 1956) for which $\theta_i^k = 0$ are characterised with the non-zero curvature tensor and exercise a mechanical influence on the detector.

Acknowledgement

I thank S. S. Gershtein and B. A. Arbuzov for discussing this paper.

References

Araki, H. (1959). *Annals of Physics* 7, 456.
Arnowitt, R., Deser, S. and Misner, C. W. (1960). *Annals of Physics,* **11**, 116.
Bauer, H. (1918). *Physikalische Zeitschrift,* 19, 163.
Brdicka, M. (1951). *Proceedings of the Royal Irish Academy,* **A54**, 137.
Brill, D. R. (1959). *Annals of Physics,* 7, 466.
Brill, D. R., Deser, S. and Fadeev, D. (1968). *Physical Review Letters,* **A26**, 538.
Brill, D. R. and Deser, S. (1968). *Physical Review Letters,* **20**, 75.
Cahen, M. and Sengier-Diels, J. (1963). *Bulletin de l'Académie & de Belgique, Classe des sciences,* **49**, 43.
Capella, M. A. (1961). *Compte rendu hebdomadaire des séances de l'Académie des sciences (Paris),* **252**, 3940.
Dirac, P. A. M. (1959). *Physical Review Letters,* **2**, 368.
De Donder, J. (1921). *La Gravifique Einsteinienne.* Paris.
Einstein, A. (1916). *Annalen dir Physik,* **49**, 769.
Einstein, A. and Rosen, N. (1937). *Journal of the Franklin Institute,* **223**, 43.
Einstein, A. (1941). *Revista. Instituto de física Universidad nacional de Tucumán,* **A2**, 11.
Einstein, A. and Pauli, W. (1943). *Annals of Mathematics,* **44**, 131.
Fok, V. A. (1961). *Theory of Space, Time, and Gravitation.* Publishing House of Phys.-Math. Literature, Moscow.

Folomeshkin, V. N. (1967). *Jadernaja Fisika,* **6,** 1080; (1968, *Soviet Journal of Nuclear Physics,* **6,** 785).
Folomeshkin, V. N. (1969). *Acta physiologica polonica,* **36,** 439.
Folomeshkin, V. N. (1970). *Acta Physiologica polonica,* **B1,** 13.
Folomeshkin, V. N. (1971). Institute of High Energy Physics Preprint IHEP 71-120 (Serpukhov).
Folomeshkin, V. N. (1972). Institute of High Energy Physics Preprint IHEP 72-92 (Serpukhov).
Freud, Ph. (1939). *Annals of Mathematics,* **40,** 417.
Havas, P. and Goldberg, J. N. (1962). *Physical Review,* **128,** 398.
Hu, N. (1947). *Proceedings of the Royal Irish Academy,* **A51,** 87.
Infeld, L. and Sheidegger, A. E. (1951). *Canadian Journal of Mathematics,* **3,** 195.
Infeld, L. (1953). *Canadian Journal of Mathematics,* **5,** 17.
Infeld, L. (1956). *Math. Ann.* **17,** 545.
Infeld, L. (1959). *Annals of Physics,* **16,** 341.
Infeld, L. and Plebansky, J. (1960). *Motion and Relativity.* Pergamon Press, New York.
Klark, G. L. (1947). *Proceedings of the Royal Society of Edinburgh,* **62,** 412.
Komar, A. (1963). *Physical Review,* **129,** 1873.
Kuchar, K. and Langer, J. (1963). *Czechoslovak Journal of Physics,* **B13,** 233.
Lanczos, K. (1923). *Physikalische Zeitschrift,* **23,** 537.
Langer, J. (1963). *Czechoslovak Journal of Physics,* **B13,** 789.
Lorentz, H. A. (1916a). *Verslagen van de gewone vergadering der wis- en natuurkundige afdeeling. K. Akademie van wetenschaffen te Amsterdam,* **25,** 468.
Lorentz, H. A. (1916b). *Verslagen van de gewone vergadering der wis- en natuurkundige afdeeling. K. Akademie van wetenschaffen te Amsterdam,* **25,** *1380.*
Misner, C. W. (1963). *Physical Review,* **130,** 1590.
Møller, Ch. (1958a). In *Max-Planck Festschrift,* p. 139. Berlin.
Møller, Ch. (1958b). *Annals of Physics,* **4,** 347.
Møller, Ch. (1964). In *Atti del Convegno sulla Relativita Generale: Problemi dell Energia e Ondre Gravitazionalli,* v. 2. Roma.
Nordström, G. (1918). *Proceedings K. Nederlandse akademie van wetenschaffen,* **20,** 1076.
Papapetrou, A. (1947). *Proceedings of the Royal Irish Academy,* **A51,** 191.
Peres, A. (1947). *Nuovo cimento,* **11,** 644.
Petrov, A. Z., Piragas, K. A. and Dobrovolsky, V. A. (1968). In *Gravitation and Relativity,* N 4–5, p. 44. Kazan, University Publishing House.
Pirani, F. A. E. (1961). In *Gravitation: Introduction to Current Research.* John Wiley and Sons, New York.
Rosen, N. (1953). *Bulletin of the Research Council of Israel,* **3,** 328.
Rosen, N. (1956). In *Jubilee of Relativity Theory,* p. 170. Birkhauser Verlag, Basel.
Ruse, H. S. (1935). *Proceedings of the Edinburgh Mathematical Society,* **4,** 144.
Schrödinger, E. (1918). *Physikalische Zeitschrift,* **19,** 4.
Sexl, R. U. (1966). *Physical Review Letters,* **20,** 376.
Scheidegger, A. E. (1951). *Physical Review,* **82,** 833.
Scheidegger, A. E. (1953). *Review of Modern Physics,* **25,** 451.
Scheidegger, A. E. (1955). *Physical Review,* **99,** 1883.
Shirokov, M. F. (1972). In *Gravitation.* Publishing House 'Naukova Dumka', Kiev.
Shirokov, M. F. (1972b). Institute of Theoretical Physics Preprint ITP-72-131E (Kiev).
Tolman, R. C. (1930). *Physical Review,* **35,** 875.
Trautman, A. (1958). *Bulletin de l'Acadimie polonaise des sciences,* **6,** 627.
Trautman, A. (1958b). In *Lectures on General Relativity.* King's College, London.
Vlasov, Yu. P. (1971). Institute of Theor. Phys. Preprint ITP-71-48 (Kiev).
Vu Thanh Khiet (1965). *Annales de l'Institut Henri Poincaré,* **2,** 261.
Weber, J. and Wheeler, J. (1957). *Review of Modern Physics,* **29,** 509.
Whittaker, E. T. (1935). *Proceedings of the Royal Society (London),* **149,** 384.